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# Elliptic hypergeometric series on root systems

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## Abstract

We derive a number of summation and transformation formulas for elliptic hypergeometric series on the root systems  $A_n$ ,  $C_n$  and  $D_n$ . In the special cases of classical and  $q$ -series, our approach leads to new elementary proofs of the corresponding identities.

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## 1. Introduction

The study of hypergeometric series has, since the days of Gauss, been an active field of research with an increasing number of applications. In a wide sense, this field includes a large number of different extensions of Gauss's hypergeometric series, such as the basic hypergeometric series or  $q$ -series first studied by Cauchy and Heine [14]. During the last 20 years,  $q$ -series have, for good reasons, been enormously popular. Apart from the intrinsic beauty of the subject, the main reasons seem to be, on the one hand, applications of such series in number theory and combinatorics, on the other hand, their relation to various integrable models in mathematical physics and to the related algebraic structures known as quantum groups.

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Recently, a new natural extension of hypergeometric series was introduced, namely, the elliptic or modular hypergeometric series of Frenkel and Turaev [13]. Subsequent papers on this topic include [7,9,35,40–43,46]. The motivation came from statistical mechanics; namely, such series may be used to express elliptic  $6j$ -symbols, which are elliptic solutions of the Yang–Baxter equation found by Date et al. [4]. Moreover, elliptic hypergeometric series are intimately connected to the elliptic gamma function [12,37], which recently appeared in connection with integrable systems. (As the referee kindly pointed out, this function can be found already in the work of Jackson [23, Section 5].) Integrable models involving elliptic functions and the related elliptic quantum groups [11] are of great current interest. It is likely that their relations to elliptic hypergeometric series go far beyond what is presently known.

From a mathematical point of view, elliptic hypergeometric series are very natural objects. Recall that a series  $\sum_k a_k$  is called hypergeometric if  $f(k) = a_{k+1}/a_k$  is a rational function of  $k$ , and a  $q$ -series if  $f$  is a rational function of  $q^k$  for a fixed  $q$ . This may be compared with Weierstrass's theorem, stating that a meromorphic function in  $z$  satisfying an algebraic addition theorem is either rational, rational in  $q^z$  (we may then write  $q = e^{ih}$  and call the function trigonometric) or an elliptic function. Elliptic hypergeometric series form the top level of the hierarchy “rational—trigonometric—elliptic” within the framework of hypergeometric series. Namely, for elliptic hypergeometric series the term ratio  $f$  is an elliptic function. (It may seem natural to, more generally, allow  $f$  to be doubly quasi-periodic, but it appears that in interesting cases it is not only elliptic but also has modular invariance; cf. [41] for a detailed discussion.)

Another natural extension of hypergeometric series is known as hypergeometric series on root systems. Loosely speaking, these are multivariable series such that the Weyl denominator of a classical root system is responsible for the coupling of the summation indices. During the last 25 years, this type of series (classical and  $q$ -) has been studied intensively by Gustafson, Milne and many other researchers. The theory has found many applications, such as to plane partitions [15,25] and to representation of integers as sums of squares [30]. Hypergeometric series on root systems are also closely related to Macdonald polynomials and other multivariable orthogonal polynomials [6,16,24,34,44] and to Selberg-type integrals and Macdonald–Morris-type identities for root systems [6,18–20,26]. (The cited references are a small number of representative papers from a large and growing literature.)

Hypergeometric series on root systems first appeared (in the case of the root system  $A_n$ ) in the work of Biedenharn et al. [22], where they were used to express multiplicity-free  $6j$ -symbols of the group  $SU(n)$ . Thus, both elliptic hypergeometric series and hypergeometric series on root systems were motivated by mathematical physics, and even more precisely by different extensions of Racah's expression for the classical  $6j$ -symbol as a  ${}_4F_3$  hypergeometric sum.

The subject of the present work is the natural unification of these two fields: *elliptic hypergeometric series on root systems*.

It should be pointed out that creating a theory of elliptic hypergeometric series on root systems is not merely a matter of plugging an extra parameter into existing works. The reason is that the theory of (one- and multivariable) hypergeometric series is usually built up as a ladder. That is, one starts with the simplest results such as the binomial theorem and Gauss's  ${}_2F_1$  sum, and use these to successively build more complicated identities. A natural culmination point is given by Jackson's  ${}_8W_7$  summation and Bailey's  ${}_{10}W_9$  transformation (or their multivariable analogues). This is a natural approach which follows the historical development, but it is doomed to fail for elliptic hypergeometric series (which is surely one reason why it took so long before these were discovered). The problem is that, for all known elliptic hypergeometric series identities, the simplest non-trivial case is the addition formula

$$\begin{aligned} & \vartheta(x+z)\vartheta(x-z)\vartheta(y+w)\vartheta(y-w) \\ &= \vartheta(x+y)\vartheta(x-y)\vartheta(z+w)\vartheta(z-w) + \vartheta(x+w)\vartheta(x-w)\vartheta(y+z)\vartheta(y-z) \end{aligned} \quad (1.1)$$

for a suitably defined theta function  $\vartheta$  (this is not an addition theorem in the sense of Weierstrass). In our figurative ladder, the lower steps correspond to lower level addition formulas such as

$$\sin(x+z)\sin(x-z) = \sin(x+y)\sin(x-y) + \sin(y+z)\sin(y-z),$$

which do not admit elliptic analogues. This means that the theory of elliptic hypergeometric series looks like a ladder with all but the top rungs missing. Thus, instead of climbing the ladder one must find a way to start at the top. This may be viewed as additional motivation for studying elliptic hypergeometric series, since such an alternative approach should add something to our understanding also of the rational and trigonometric levels.

The present paper is not the first to deal with elliptic hypergeometric series on root systems. Warnaar [46] proved one multivariable elliptic analogue of Jackson's  ${}_8W_7$  summation formula (cf. (7.1) below) and conjectured another one. We will refer to these as Warnaar's first and second summation. Van Diejen and Spiridonov [7] found that the second summation may be derived from a certain conjectured elliptic Selberg integral. In [35], we gave an inductive proof of Warnaar's second summation, using a very special case of his first summation in the inductive step. It should be said that, although both Warnaar sums live on root systems (to be precise, on  $C_n$ ), they are somewhat different in nature from those going back to the seminal paper [22]. In the terminology of [9], the first sum is of "Schlosser-type" and the second one of "Aomoto–Itô–Macdonald-type", as opposed to the Gustafson–Milne-type series which are our main concern here. In the latter direction, van Diejen and Spiridonov [9] stated a third elliptic  $C_n$  analogue of Jackson's summation formula (cf. Theorem 7.1 below), which in the trigonometric case is due independently to Denis and Gustafson [5] and to Milne and Lilly [31]. However, this identity was only proved under the condition that a certain Selberg-type integral evaluation conjectured in [8] holds.

The purpose of the present paper is to give elliptic analogues of a number of known identities for Gustafson–Milne-type series on the root systems  $A_n$ ,  $C_n$  and  $D_n$ . We have focused on deriving some of the most fundamental and important identities—analogue of Jackson’s  ${}_8W_7$  summation and Bailey’s  ${}_{10}W_9$  transformation, although it seems likely that more identities (for instance, quadratic and bibasic summation formulas) may be derived by our methods. In the one-variable case, our identities reduce to either the elliptic Jackson summation or the elliptic Bailey transformation of Frenkel and Turaev [13].

To obtain elliptic Jackson summations on the root systems  $A_n$  and  $D_n$  we use an inductive method similar to our proof of Warnaar’s second summation in [35]. Instead of applying Warnaar’s first summation in the induction step, we use elliptic analogues of elementary partial fraction expansions. In our opinion, the resulting proofs are simpler than those previously known for classical and  $q$ -series, even for very degenerate cases of the identities. For the root system  $C_n$ , we cannot use this approach. Instead, we observe that the elliptic  $C_n$  Jackson summation we want to prove (the same one that was conjectured in [9]) may be obtained, in a rather non-obvious way, as a special case of Warnaar’s first summation. This gives an unexpected link with hypergeometric series connected with determinant evaluations (the Schlosser-type series alluded to above).

The plan of the paper is as follows. In Section 2, we give a first illustration of our approach to  $A_n$  and  $D_n$  series, by showing how it works in a simple case, Milne’s  $A_n$  analogue of the terminating  ${}_6W_5$  summation. Sections 3 and 4 contain notation and preliminaries for theta functions and elliptic hypergeometric series. Our main results are contained in Sections 5–7, where we derive elliptic Jackson summations on the root systems  $A_n$ ,  $D_n$  and  $C_n$ , respectively. In Section 8 we apply these summations to derive a number of elliptic Bailey transformations on these root systems. Since the step from summation formulas to transformation formulas works exactly as in the rational and trigonometric case, we do not provide much details in this final section.

## 2. Milne’s fundamental theorem

To illustrate our method, we start with one of the simplest identities for hypergeometric series on root systems, Milne’s fundamental theorem of  $U(n)$  series. *In this section only* we use the notation

$$(a)_k = (1-a)(1-aq)\cdots(1-aq^{k-1}), \quad (2.1)$$

where  $q$  is a fixed parameter suppressed from the notation. We also write, *again in this section only*,

$$A(z) = \prod_{1 \leq j < k \leq n} (z_j - z_k)$$

for the Vandermonde determinant, which is the Weyl denominator for the root system  $A_n$ . Basic hypergeometric series on  $A_n$  contain the factor

$$\frac{\Delta(zq^y)}{\Delta(z)} = \prod_{1 \leq j < k \leq n} \frac{z_j q^{y_j} - z_k q^{y_k}}{z_j - z_k},$$

where the  $z_j$  are fixed parameters and the  $y_j$  summation indices.

We can now state Milne's fundamental theorem:

$$\sum_{\substack{y_1, \dots, y_n \geq 0 \\ y_1 + \dots + y_n = N}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j,k=1}^n \frac{(a_j z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} = \frac{(a_1 \cdots a_n)_N}{(q)_N}. \quad (2.2)$$

When  $n = 2$  this is Rogers's terminating  ${}_6W_5$  summation formula [14, Eq. (II.20)]. On the other hand, multiplying (2.2) with  $t^N$  and then summing over non-negative integers  $N$  gives an extension of Cauchy's  $q$ -binomial theorem [14, Eq. (II.3)]. So one may also view (2.2) as a refined multivariable  $q$ -binomial theorem.

Identity (2.2) was first obtained in [26], where it was used to derive the Macdonald identities for the affine Lie algebra  $A_n^{(1)}$ . In Milne's approach to hypergeometric series on  $A_n$  or  $U(n)$  (cf. [29] for a recent survey), (2.2) is the starting point from which all other results are built; this motivates the name “fundamental theorem of  $U(n)$  series” [1]. By contrast, in the present approach (2.2) appears as a degenerate case of the elliptic Jackson summation given in Theorem 5.1 below, which we will prove by precisely the same method. We only give (2.2) a separate treatment in order to illustrate our method in the simplest possible situation.

Milne's proof of (2.2) is based on showing that both sides satisfy the same difference equation in the variables  $a_k$  and  $z_k$ . An important tool is an identity which we write here as

$$\sum_{k=1}^n \frac{\prod_{j=1}^n (b_j - a_k)}{a_k \prod_{j \neq k} (a_j - a_k)} = \frac{b_1 \cdots b_n}{a_1 \cdots a_n} - 1 \quad (2.3)$$

(throughout the paper we tacitly assume that all expressions are well-defined, in this case that the  $a_j$  are distinct and non-zero). Our proof is also based on (2.3), but instead of difference operators we use a simple inductive argument.

One way to prove (2.3) is to rewrite it as a partial fraction expansion, replacing  $a_j$ ,  $b_j$ , by  $a_j - t$ ,  $b_j - t$ :

$$\prod_{j=1}^n \frac{b_j - t}{a_j - t} = 1 + \sum_{k=1}^n \frac{\prod_{j=1}^n (b_j - a_k)}{(a_k - t) \prod_{j \neq k} (a_j - a_k)}.$$

The proof is then easy: the existence of the expansion is immediate by induction on  $n$ , and to compute the coefficients one may multiply with  $a_k - t$  and plug in  $t = a_k$ . This is taught at Swedish technical universities under the name “handpåläggnig” (laying on hands) [33].

We prove (2.2) by induction on  $N$ , starting with  $N = 1$ . In that case we have  $y_i = \delta_{ik}$  for some  $k$ . Using  $k$  as summation index, the left-hand side of (2.2) may be rewritten and summed using (2.3):

$$\sum_{k=1}^n \prod_{j \neq k} \frac{z_j - qz_k}{z_j - z_k} \prod_{j=1}^n \frac{z_j - a_j z_k}{z_j - qz_k} = \frac{a_1 \cdots a_n}{1 - q} \sum_{k=1}^n \frac{\prod_{j=1}^n (z_j/a_j - z_k)}{z_k \prod_{j \neq k} (z_j - z_k)} = \frac{1 - a_1 \cdots a_n}{1 - q}.$$

In fact, we see that the case  $N = 1$  of (2.2) is equivalent to (2.3).

Next we assume that (2.2) holds for a fixed value of  $N$ . We may then write the right-hand side, with  $N$  replaced by  $N + 1$ , as

$$\begin{aligned} R &= \frac{(a_1 \cdots a_n)_{N+1}}{(q)_{N+1}} = \frac{1 - q^N a_1 \cdots a_n}{1 - q^{N+1}} \frac{(a_1 \cdots a_n)_N}{(q)_N} \\ &= \sum_{\substack{y_1, \dots, y_n \geq 0 \\ y_1 + \dots + y_n = N}} \frac{1 - q^N a_1 \cdots a_n}{1 - q^{N+1}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j,k=1}^n \frac{(a_j z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}}. \end{aligned}$$

We now split the terms using the case  $N = 1$  of (2.2), with  $a_k$  replaced by  $a_k q^{y_k}$  and  $z_k$  by  $z_k q^{y_k}$ . Explicitly, we have

$$\frac{1 - q^N a_1 \cdots a_n}{1 - q} = \sum_{\substack{x_1, \dots, x_n \geq 0 \\ x_1 + \dots + x_n = 1}} \frac{\Delta(zq^{y+x})}{\Delta(zq^y)} \prod_{j,k=1}^n \frac{(q^{y_k} a_j z_k / z_j)_{x_k}}{(q^{1+y_k-y_j} z_k / z_j)_{x_k}},$$

which gives

$$\begin{aligned} R &= \frac{1 - q}{1 - q^{N+1}} \sum_{\substack{y_1, \dots, y_n \geq 0 \\ y_1 + \dots + y_n = N}} \sum_{\substack{x_1, \dots, x_n \geq 0 \\ x_1 + \dots + x_n = 1}} \frac{\Delta(zq^{y+x})}{\Delta(z)} \prod_{j,k=1}^n \frac{(a_j z_k / z_j)_{y_k + x_k}}{(q z_k / z_j)_{y_k} (q^{1+y_k-y_j} z_k / z_j)_{x_k}} \\ &= \frac{1 - q}{1 - q^{N+1}} \sum_{\substack{y_1, \dots, y_n \geq 0 \\ y_1 + \dots + y_n = N+1}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j,k=1}^n (a_j z_k / z_j)_{y_k} \\ &\quad \times \sum_{\substack{0 \leq x_k \leq y_k \\ x_1 + \dots + x_n = 1}} \prod_{j,k=1}^n \frac{1}{(q z_k / z_j)_{y_k - x_k} (q^{1+y_k-y_j-x_k+x_j} z_k / z_j)_{x_k}}, \end{aligned} \quad (2.4)$$

where we replaced  $y$  by  $y - x$ .

We will identify the sum in  $x$  as a special case of (2.3). Since  $x_k \in \{0, 1\}$ , we have

$$\frac{1}{(q z_k / z_j)_{y_k - x_k}} = \frac{(q^{1+y_k-x_k} z_k / z_j)_{x_k}}{(q z_k / z_j)_{y_k}} = \frac{(q^{y_k} z_k / z_j)_{x_k}}{(q z_k / z_j)_{y_k}}. \quad (2.5)$$

Writing the factor in that form makes the conditions  $x_k \leq y_k$  superfluous, since  $(q^{y_k})_{x_k}$  vanishes if  $y_k = 0$  and  $x_k = 1$ . Moreover, if  $j \neq k$  the factor

$(q^{1+y_k-y_j-x_k+x_j} z_k/z_j)_{x_k}$  equals 1 unless  $x_k = 1$  and  $x_j = 0$ , which gives

$$\prod_{j,k=1}^n (q^{1+y_k-y_j-x_k+x_j} z_k/z_j)_{x_k} = (1-q) \prod_{j \neq k} (q^{y_k-y_j} z_k/z_j)_{x_k}. \quad (2.6)$$

We may now compute the sum in  $x$  using (2.3):

$$\begin{aligned} & \frac{1}{1-q} \prod_{j,k=1}^n \frac{1}{(qz_k/z_j)_{y_k}} \sum_{\substack{x_1, \dots, x_n \geq 0 \\ x_1 + \dots + x_n = 1}} \frac{\prod_{j,k=1}^n (q^{y_k} z_k/z_j)_{x_k}}{\prod_{j \neq k} (q^{y_k-y_j} z_k/z_j)_{x_k}} \\ &= \frac{q^{N+1}}{1-q} \prod_{j,k=1}^n \frac{1}{(qz_k/z_j)_{y_k}} \sum_{k=1}^n \frac{\prod_{j=1}^n (z_j - q^{y_k} z_k)}{q^{y_k} z_k \prod_{j \neq k} (q^{y_j} z_j - q^{y_k} z_k)} \\ &= \frac{1-q^{N+1}}{1-q} \prod_{j,k=1}^n \frac{1}{(qz_k/z_j)_{y_k}}. \end{aligned}$$

Plugging this into (2.4) gives

$$R = \sum_{\substack{y_1, \dots, y_n \geq 0 \\ y_1 + \dots + y_n = N+1}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j,k=1}^n \frac{(a_j z_k/z_j)_{y_k}}{(qz_k/z_j)_{y_k}},$$

which is the left-hand side of (2.2) with  $N$  replaced by  $N+1$ . This completes our proof of Milne's fundamental theorem.

### 3. Notation and preliminaries

Elliptic hypergeometric series may be built from the theta function

$$\theta(x) = \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x), \quad |p| < 1,$$

where  $p$  is a constant which is fixed throughout the paper. It satisfies the inversion formula

$$\theta(x) = -x\theta(1/x), \quad (3.1)$$

the quasi-periodicity

$$\theta(px) = -\frac{1}{x} \theta(x) \quad (3.2)$$

and the “addition formula”

$$\begin{aligned} & \theta(xz) \theta(x/z) \theta(yw) \theta(y/w) \\ &= \frac{y}{z} \theta(xy) \theta(x/y) \theta(zw) \theta(z/w) + \theta(xw) \theta(x/w) \theta(yz) \theta(y/z) \end{aligned} \quad (3.3)$$

(cf. [14, Exercises 2.16 and 5.21] for proofs within the framework of basic hypergeometric series). Equivalently,  $\vartheta(x) = q^{-x/2} \theta(q^x)$ , with  $q$  fixed, satisfies (1.1). All results of this paper may be viewed as generalizations of (1.1).

We define elliptic Pochhammer symbols by

$$(a)_k = \theta(a) \theta(aq) \cdots \theta(aq^{k-1}).$$

Note that they depend on two parameters  $p, q$ , which will be suppressed from the notation. When  $p = 0$ ,  $\theta(x) = 1 - x$  and we recover the symbols (2.1) used in the previous section. The elliptic symbols satisfy similar elementary identities as in the case  $p = 0$ , such as

$$(a)_{n+k} = (a)_n (aq^n)_k, \quad (3.4)$$

$$(a)_{n-k} = (-1)^k q^{\binom{k}{2}} (q^{1-n}/a)^k \frac{(a)_n}{(q^{1-n}/a)_k}, \quad (3.5)$$

$$(a)_n = (-1)^n q^{\binom{n}{2}} a^n (q^{1-n}/a)_n; \quad (3.6)$$

these will be used repeatedly and usually without comment. We also need the quasi-periodicity

$$(pa)_k = (-1)^k q^{-\binom{k}{2}} a^{-k} (a)_k, \quad (3.7)$$

which follows from (3.2).

We introduce the elliptic Vandermonde determinant

$$\Delta(z) = \prod_{1 \leq j < k \leq n} z_j \theta(z_k/z_j),$$

which appears in elliptic hypergeometric series on  $A_n$  and, together with additional double products, in the series on  $C_n$  and  $D_n$ .

The following identity will be useful (cf. [28, Lemma 4.3] for a more general identity in the case  $p = 0$ ):

$$(-1)^{|y|} q^{|y| + \binom{|y|}{2}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j,k=1}^n \frac{(q^{-y_j} z_k/z_j)_{y_k}}{(qz_k/z_j)_{y_k}} = 1. \quad (3.8)$$



To prove this, one may rewrite the double product as

$$\prod_{j,k=1}^n \frac{(q^{-y_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} = \prod_{k=1}^n \frac{(q^{-y_k})_{y_k}}{(q)_{y_k}} \prod_{1 \leq j < k \leq n} \frac{(q^{-y_j} z_k / z_j)_{y_k} (q^{-y_k} z_j / z_k)_{y_j}}{(q z_k / z_j)_{y_k} (q z_j / z_k)_{y_j}},$$

which, by (3.6), equals

$$\begin{aligned} & \prod_{k=1}^n (-1)^{y_k} q^{\binom{y_k}{2} - y_k^2} \prod_{1 \leq j < k \leq n} q^{-(y_k+1)y_j} \frac{(q^{-y_j} z_k / z_j)_{y_k} (q^{1-y_j+y_k} z_k / z_j)_{y_j}}{(q z_k / z_j)_{y_k} (q^{-y_j} z_k / z_j)_{y_j}} \\ &= (-1)^{|y|} q^{\sum_k \binom{y_k}{2} - y_k^2} q^{-\sum_{j < k} y_k y_j} \prod_{1 \leq j < k \leq n} q^{-y_j} \frac{\theta(z_k / z_j)}{\theta(q^{y_k - y_j} z_k / z_j)} \\ &= (-1)^{|y|} q^{-\binom{|y|}{2} - |y|} \frac{\Delta(z)}{\Delta(z q^y)}. \end{aligned}$$

Finally, we will often use the shorthand notation

$$\theta(a_1, \dots, a_n) = \theta(a_1) \cdots \theta(a_n),$$

$$(a_1, \dots, a_n)_k = (a_1)_k \cdots (a_n)_k.$$

#### 4. Elliptic partial fraction expansions

The starting point for our investigation of elliptic hypergeometric series on  $A_n$  will be the following elliptic generalization of (2.3):

$$\sum_{k=1}^n \frac{\prod_{j=1}^n \theta(a_k / b_j)}{\prod_{j \neq k} \theta(a_k / a_j)} = 0, \quad a_1 \cdots a_n = b_1 \cdots b_n, \quad (4.1)$$

which is valid as long as both sides are well-defined. We do not know who first wrote down this identity. It may be found in the 1898 treatise of Tannery and Molk [45, p. 34] and, as an exercise, in the classic textbook of Whittaker and Watson [47, p. 451].

To see that (4.1) generalizes (2.3), replace  $n$  by  $n+1$  and write  $a_{n+1} = t$ . Isolating the term with  $k = n+1$ , (4.1) may be written as

$$\sum_{k=1}^n \frac{\prod_{j=1}^{n+1} \theta(a_k / b_j)}{\theta(a_k / t) \prod_{j \neq k} \theta(a_k / a_j)} = - \frac{\prod_{j=1}^{n+1} \theta(t / b_j)}{\prod_{j=1}^n \theta(t / a_j)}, \quad b_1 \cdots b_{n+1} = a_1 \cdots a_n t \quad (4.2)$$

(note that one may recover (4.1) also by letting  $t = b_{n+1}$  in (4.2)). If we put  $p = 0$  and then let  $t, b_{n+1} \rightarrow 0$  with  $t/b_{n+1} = b_1 \cdots b_n / a_1 \cdots a_n$  fixed we recover (2.3). We also

observe that, by (3.1), (4.2) may alternatively be written as

$$\sum_{k=1}^n \frac{\prod_{j=1}^{n+1} \theta(a_k/b_j)}{\theta(a_k/t) \prod_{j \neq k} \theta(a_k/a_j)} = \frac{\prod_{j=1}^{n+1} \theta(b_j/t)}{\prod_{j=1}^n \theta(a_j/t)}, \quad b_1 \cdots b_{n+1} = a_1 \cdots a_n t. \quad (4.3)$$

A simple proof of (4.2) arises from the interpretation as a generalized partial fraction expansion. Namely, for  $n = 2$ , (4.2) is equivalent to the addition formula (3.3). It then follows immediately by induction on  $n$  that there exists an expansion

$$\prod_{j=1}^n \frac{\theta(t/b_j)}{\theta(t/a_j)} = \sum_{k=1}^n C_k \frac{\theta(a_k b_1 \cdots b_n / a_1 \cdots a_n t)}{\theta(a_k/t)}, \quad (4.4)$$

where the coefficients are easily computed using “handpåläggning”. That is, if one multiplies (4.4) with  $\theta(t/a_k) = -\theta(a_k/t)t/a_k$  and let  $t = a_k$  one obtains

$$C_k = -\frac{\prod_{j=1}^n \theta(a_k/b_j)}{\theta(b_1 \cdots b_n / a_1 \cdots a_n) \prod_{j \neq k} \theta(a_k/a_j)}.$$

Plugging this into (4.4) and writing  $b_{n+1} = a_1 \cdots a_n t / b_1 \cdots b_n$  yields (4.2).

For the root system  $D_n$  we need in addition to (4.1) the identity

$$\sum_{k=1}^n \frac{\prod_{j=1}^{n-1} \theta(a_k b_j, a_k/b_j)}{\theta(t a_k, t/a_k) \prod_{j \neq k} \theta(a_k a_j, a_k/a_j)} = \frac{\prod_{j=1}^{n-1} \theta(t b_j, t/b_j)}{\prod_{j=1}^n \theta(t a_j, t/a_j)}, \quad (4.5)$$

or, equivalently,

$$\sum_{k=1}^n \frac{a_k \prod_{j=1}^{n-2} \theta(a_k b_j, a_k/b_j)}{\prod_{j \neq k} \theta(a_k a_j, a_k/a_j)} = 0. \quad (4.6)$$

The earliest occurrence of (4.5) that we have found is as [17, Lemma 4.14], where an analytic proof is given. Again, the interpretation as a generalized partial fraction expansion makes an algebraic proof easy: for  $n = 2$  (4.5) is equivalent to (3.3), the existence of such an expansion follows by induction on  $n$  and the coefficients may be obtained by multiplying with  $\theta(t a_k, t/a_k)$  and then letting  $t = a_k$ .

Finally, we point out some occurrences of the identities (4.1) and (4.6) in the related recent literature. In [8], they are given as Lemmas A.2 and A.1, and, written in the alternative form (4.2) and (4.5), as Propositions A.4 and A.3. Implicitly, (4.6) also appears in [46]. Namely, in [46, Lemma 3.2] a certain matrix inversion, say  $AB = I$ , is obtained as a special case of a more general result. The inverse relation  $BA = I$  is easily seen to be equivalent to (4.6).

## 5. An elliptic $A_n$ Jackson summation

The first main result of the paper is the following Jackson-type summation formula for an elliptic hypergeometric series on the root system  $A_n$ . When  $p = 0$  we recover (with a new, elementary proof) Milne's  $A_n$  Jackson summation [27, Theorem 6.17]. To see this, one should rewrite our identity as in Corollary 5.2 below and Milne's identity as in [32, Theorem A.5]. Note also that (2.2) may be obtained from the case  $p = 0$  of Theorem 5.1 by letting  $b, a_{n+1} \rightarrow 0$  with  $b/a_{n+1}$  fixed.

**Theorem 5.1.** *If  $b = a_1 \cdots a_{n+1} z_1 \cdots z_n$ , the following identity holds:*

$$\sum_{\substack{y_1 + \cdots + y_n = N \\ y_1, \dots, y_n \geq 0}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\prod_{j=1}^{n+1} (a_j z_k)_{y_k}}{(bz_k)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}} = \frac{(b/a_1, \dots, b/a_{n+1})_N}{(q, bz_1, \dots, bz_n)_N}.$$

Note that Theorem 5.1 contains one superfluous parameter. For instance,  $b$  may be scaled away by replacing  $z_k \mapsto z_k/b$ ,  $a_j \mapsto a_j b$ . This is true for most of our results. We have chosen to introduce superfluous parameters to facilitate comparison with related results in the literature, and also to make the structure of the identities more transparent.

**Proof of Theorem 5.1.** We will use induction on  $N$ . If  $N = 1$ , we have  $y_i = \delta_{ik}$  for some  $k$ ; using  $k$  as summation variable yields the identity

$$\sum_{k=1}^n \frac{\prod_{j=1}^{n+1} \theta(a_j z_k)}{\theta(bz_k) \prod_{j \neq k} \theta(z_k/z_j)} = \frac{\prod_{j=1}^{n+1} \theta(b/a_j)}{\prod_{j=1}^n \theta(bz_j)}, \quad b = a_1 \cdots a_{n+1} z_1 \cdots z_n,$$

which is equivalent to (4.3).

Assume that the theorem holds for a fixed  $N$ , and let

$$R = \frac{(b/a_1, \dots, b/a_{n+1})_{N+1}}{(q, bz_1, \dots, bz_n)_{N+1}}.$$

Imitating the proof of (2.2) given above, we rewrite  $R$  using the induction hypothesis, and then expand each term using the case  $N = 1$ ,  $z_k \mapsto q^{y_k} z_k$ ,  $b \mapsto q^N b$  of the theorem. Explicitly, this gives

$$\begin{aligned} R &= \frac{\prod_{k=1}^{n+1} \theta(q^N b/a_k)}{\theta(q^{N+1}) \prod_{k=1}^n \theta(q^N bz_k)} \frac{(b/a_1, \dots, b/a_{n+1})_N}{(q, bz_1, \dots, bz_n)_N} \\ &= \frac{\prod_{k=1}^{n+1} \theta(q^N b/a_k)}{\theta(q^{N+1}) \prod_{k=1}^n \theta(q^N bz_k)} \sum_{\substack{y_1 + \cdots + y_n = N \\ y_1, \dots, y_n \geq 0}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\prod_{j=1}^{n+1} (a_j z_k)_{y_k}}{(bz_k)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}} \end{aligned}$$

$$\begin{aligned}
&= \frac{\theta(q)}{\theta(q^{N+1})} \sum_{\substack{y_1+\dots+y_n=N \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(q^{N+y_k}bz_k) \prod_{j=1}^{n+1} (a_j z_k)_{y_k}}{\theta(q^N bz_k)(bz_k)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}} \right. \\
&\quad \times \left. \sum_{\substack{x_1+\dots+x_n=1 \\ x_1, \dots, x_n \geq 0}} \frac{\Delta(zq^{y+x})}{\Delta(zq^y)} \prod_{k=1}^n \frac{\prod_{j=1}^{n+1} (q^{y_k} a_j z_k)_{x_k}}{(q^{N+y_k}bz_k)_{x_k} \prod_{j=1}^n (q^{1+y_k-y_j}z_k/z_j)_{x_k}} \right).
\end{aligned}$$

Next we replace  $y$  by  $y - x$  in the summation. Since  $x_k \in \{0, 1\}$ , we may write

$$\begin{aligned}
\frac{\theta(q^{N+y_k-x_k}bz_k)}{(q^{N+y_k-x_k}bz_k)_{x_k}} &= \frac{\theta(q^{N+y_k}bz_k)}{(q^{N+y_k}bz_k)_{x_k}}, \\
\frac{1}{(bz_k)_{y_k-x_k}} &= \frac{(q^{y_k-1}bz_k)_{x_k}}{(bz_k)_{y_k}}.
\end{aligned}$$

It is clear that (2.5) holds also in the elliptic case, and that (2.6) has the elliptic analogue

$$\prod_{j,k=1}^n (q^{1+y_k-y_j-x_k+x_j}z_k/z_j)_{x_k} = \theta(q) \prod_{j \neq k} (q^{y_k-y_j}z_k/z_j)_{x_k}.$$

This gives the expression

$$\begin{aligned}
R &= \frac{1}{\theta(q^{N+1})} \sum_{\substack{y_1+\dots+y_n=N+1 \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(q^{N+y_k}bz_k) \prod_{j=1}^{n+1} (a_j z_k)_{y_k}}{\theta(q^N bz_k)(bz_k)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}} \right. \\
&\quad \times \left. \sum_{\substack{x_1+\dots+x_n=1 \\ x_1, \dots, x_n \geq 0}} \prod_{k=1}^n \frac{(q^{y_k-1}bz_k)_{x_k} \prod_{j=1}^n (q^{y_k}z_k/z_j)_{x_k}}{(q^{N+y_k}bz_k)_{x_k} \prod_{j \neq k} (q^{y_k-y_j}z_k/z_j)_{x_k}} \right). \quad (5.1)
\end{aligned}$$

(Formally, we have introduced extra terms corresponding to  $x_k = 1$  and  $y_k = 0$ , however, these all vanish in view of the factor  $(q^{y_k}z_k/z_j)_{x_k}$ .) The sum in  $x$  may be rewritten and summed using (4.3) as

$$\sum_{k=1}^n \frac{\theta(q^{y_k-1}bz_k) \prod_{j=1}^n \theta(q^{y_k}z_k/z_j)}{\theta(q^{N+y_k}bz_k) \prod_{j \neq k} \theta(q^{y_k-y_j}z_k/z_j)} = \frac{\theta(q^{N+1}) \prod_{k=1}^n \theta(q^N bz_k)}{\prod_{k=1}^n \theta(q^{N+y_k}bz_k)}. \quad (5.2)$$

It follows that

$$R = \sum_{\substack{y_1+\dots+y_n=N+1 \\ y_1, \dots, y_n \geq 0}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\prod_{j=1}^{n+1} (a_j z_k)_{y_k}}{(bz_k)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}},$$

which completes the proof of the theorem.  $\square$

For some purposes it is convenient to rewrite Theorem 5.1 in a way that hides much of its symmetry but is closer to how the one-variable Jackson sum is normally

written. Namely, we replace  $n$  in Theorem 5.1 by  $n+1$ , eliminate  $y_{n+1} = N - (y_1 + \dots + y_n)$  from the summation and write  $z_{n+1} = a^{-1}q^{-N}$ . Then the factor  $\Delta(zq^y)/\Delta(z)$  is replaced by

$$\frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \left( q^{N-|y|} \frac{\theta(z_k q^{y_k}/z_{n+1} q^{N-|y|})}{\theta(z_k/z_{n+1})} \right) = \frac{\Delta(zq^y)}{\Delta(z)} q^{n(N-|y|)} \prod_{k=1}^n \frac{\theta(aq^{y_k+|y|}z_k)}{\theta(aq^N z_k)}.$$

After changing variables  $a_j \mapsto b_j$ ,  $b \mapsto aq/c$  and using (3.5) repeatedly, one obtains the following result.

**Corollary 5.2.** *If  $a^2 q^{1+N} = b_1 \dots b_{n+2} c z_1 \dots z_n$ , the following identity holds:*

$$\begin{aligned} & \sum_{\substack{y_1 + \dots + y_n \leq N \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \frac{(q^{-N}, c)_{|y|}}{\prod_{j=1}^{n+2} (aq/b_j)_{|y|}} \prod_{j=1}^n (az_j)_{|y|} q^{|y|} \right. \\ & \times \prod_{k=1}^n \frac{\prod_{j=1}^{n+2} (b_j z_k)_{y_k}}{(aq^{1+N} z_k, aq z_k/c)_{y_k} \prod_{j=1}^n (q z_k/z_j)_{y_k}} \Bigg) \\ & = c^N \prod_{k=1}^n \frac{(aq z_k)_N}{(aq z_k/c)_N} \prod_{k=1}^{n+2} \frac{(aq/c b_k)_N}{(aq/b_k)_N}. \end{aligned}$$

By a continuation argument, it is possible to convert Theorem 5.1 to a sum on a hyper-rectangle. When  $p = 0$ , this is [27, Theorem 6.14], given in notation more similar to ours as [32, Theorem A.5]. The step from the simplex to the hyper-rectangle is then based on polynomial continuation (two polynomials that agree in an infinite number of points are identical). In the elliptic case, we use the quasi-periodicity of the theta function to pass from an infinite set to a set having a limit point. (The same argument occurs in [46].)

**Corollary 5.3.** *If  $a^2 q^{1+|m|} = bcde$ , the following identity holds:*

$$\begin{aligned} & \sum_{\substack{m_1, \dots, m_n \\ y_1, \dots, y_n = 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \frac{(b, c)_{|y|}}{(aq/d, aq/e)_{|y|}} \prod_{j=1}^n \frac{(az_j)_{|y|}}{(aq^{1+m_j} z_j)_{|y|}} q^{|y|} 0.5pc \right. \\ & \times \prod_{k=1}^n \frac{(dz_k, ez_k)_{y_k}}{(aq z_k/b, aq z_k/c)_{y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k/z_j)_{y_k}}{(q z_k/z_j)_{y_k}} \Bigg) 0.5pc \\ & = \frac{(aq/cd, aq/bd)_{|m|}}{(aq/d, aq/bcd)_{|m|}} \prod_{k=1}^n \frac{(aq z_k, aq z_k/bc)_{m_k}}{(aq z_k/b, aq z_k/c)_{m_k}}. \end{aligned} \tag{5.3}$$

**Proof.** It is straightforward to check that the case  $b = q^{-N}$  of Corollary 5.3 is equivalent to the case  $b_j = q^{-m_j}/z_j$ ,  $1 \leq j \leq n$ , of Corollary 5.2. Consider the function

$$f(b) = \prod_{k=1}^n (aqz_k/b, aqz_k/c)_{m_k} (L - R),$$

where  $L$  and  $R$  denote the left- and right-hand sides of (5.3), respectively, and where we view  $c = a^2 q^{1+|m|}/bde$  as depending on  $b$  while the other parameters are fixed. Then  $f(b)$  is analytic for  $b \neq 0$ , and zero for  $b = q^{-N}$ ,  $N \in \mathbb{N}$ . Since, by (3.7), we have that in general

$$(\lambda pb, \lambda c/p)_k = (c/pb)^k (\lambda b, \lambda c)_k,$$

$f$  is quasi-periodic in the sense that

$$f(pb) = (c/pb)^{|m|} f(b).$$

It follows that the points  $b = p^k q^{-l}$ ,  $k \in \mathbb{Z}$ ,  $l \in \mathbb{N}$ , are zeroes of  $f$ . Assuming that these points are all distinct, that is, that  $p^k \neq q^l$  for  $k, l \in \mathbb{Z}$ , they have a limit point in  $\mathbb{C} \setminus \{0\}$  (indeed, in any annulus of the form  $\{z; pr \leq |z| \leq r\}$ ) and we may conclude that  $f \equiv 0$ . Since  $f$  depends analytically on  $p$  and  $q$ , this is true also if  $p^k = q^l$ , as long as both sides of (5.3) are well-defined.  $\square$

## 6. An elliptic $D_n$ Jackson summation

In this section we give a Jackson-type summation formula for an elliptic hypergeometric series on the root system  $D_n$ . In the case  $p = 0$ , it is due to Schlosser [38, Theorem 5.17]. As is discussed below, an essentially equivalent identity was independently found (still for  $p = 0$ ) by Bhatnagar [2].

**Theorem 6.1.** *The following identity holds:*

$$\begin{aligned} \sum_{\substack{y_1 + \dots + y_n = N \\ y_1, \dots, y_n \geq 0}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j + y_k}} \prod_{k=1}^n \frac{q^{\binom{y_k}{2}} z_k^{y_k} \prod_{j=1}^{n-1} (z_k a_j, z_k/a_j)_{y_k}}{(bz_k, q^{1-N} z_k/b)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}} \\ = (-q^{N-1} b)^N \frac{\prod_{k=1}^{n-1} (ba_k, b/a_k)_N}{(q)_N \prod_{k=1}^n (bz_k, b/z_k)_N}. \end{aligned} \quad (6.1)$$

**Proof.** The proof is by induction on  $N$ , similar to that of Theorem 5.1.

One easily checks that the case  $N = 1$  of (6.1) is equivalent to (4.5). Assume that (6.1) holds for a fixed  $N$ . Writing  $R$  for the right-hand side of (6.1) when  $N$  is replaced by  $N + 1$ , we expand  $R$  using first the induction hypothesis and then the case  $N = 1$  of (6.1) with  $z_k$  replaced with  $z_k q^{y_k}$  and  $b$  with  $q^N b$ . This gives

$$\begin{aligned}
 R &= (-q^N b)^{N+1} \frac{\prod_{k=1}^{n-1} (ba_k, b/a_k)_{N+1}}{(q)_{N+1} \prod_{k=1}^n (bz_k, b/z_k)_{N+1}} \\
 &= -q^{2N} b \frac{\prod_{k=1}^{n-1} \theta(q^N ba_k, q^N b/a_k)}{\theta(q^{N+1}) \prod_{k=1}^n \theta(q^N bz_k, q^N b/z_k)} (-q^{N-1} b)^N \frac{\prod_{k=1}^{n-1} (ba_k, b/a_k)_N}{(q)_N \prod_{k=1}^n (bz_k, b/z_k)_N} \\
 &= -q^{2N} b \frac{\prod_{k=1}^{n-1} \theta(q^N ba_k, q^N b/a_k)}{\theta(q^{N+1}) \prod_{k=1}^n \theta(q^N bz_k, q^N b/z_k)} \\
 &\quad \times \sum_{\substack{y_1 + \dots + y_n = N \\ y_1, \dots, y_n \geq 0}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j + y_k}} \prod_{k=1}^n \frac{q^{\binom{y_k}{2}} z_k^{y_k} \prod_{j=1}^{n-1} (z_k a_j, z_k/a_j)_{y_k}}{(bz_k, q^{1-N} z_k/b)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}} \\
 &= q^N \frac{\theta(q)}{\theta(q^{N+1})} \sum_{\substack{y_1 + \dots + y_n = N \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j + y_k}} \right. \\
 &\quad \times \prod_{k=1}^n \frac{\theta(q^{N+y_k} bz_k, q^{N-y_k} b/z_k) q^{\binom{y_k}{2}} z_k^{y_k} \prod_{j=1}^{n-1} (z_k a_j, z_k/a_j)_{y_k}}{\theta(q^N bz_k, q^N b/z_k) (bz_k, q^{1-N} z_k/b)_{y_k} \prod_{j=1}^n (qz_k/z_j)_{y_k}} \\
 &\quad \times \sum_{\substack{x_1 + \dots + x_n = 1 \\ x_1, \dots, x_n \geq 0}} \left\{ \frac{\Delta(zq^{y+x})}{\Delta(zq^y)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k q^{y_j + y_k})_{x_j + x_k}} \right. \\
 &\quad \times \left. \left. \prod_{k=1}^n \frac{q^{\binom{x_k}{2} + y_k x_k} z_k^{x_k} \prod_{j=1}^{n-1} (z_k q^{y_k} a_j, z_k q^{y_k}/a_j)_{x_k}}{(q^{N+y_k} bz_k, q^{y_k-N} z_k/b)_{x_k} \prod_{j=1}^n (q^{1+y_k-y_j} z_k/z_j)_{x_k}} \right\} \right).
 \end{aligned}$$

Next we replace  $y$  by  $y - x$  and perform the same simplifications as in the proof of Theorem 5.1. We also write

$$\frac{\theta(q^{N-y_k+x_k} b/z_k)}{\theta(q^N b/z_k)} \frac{1}{(q^{1-N} z_k/b)_{y_k-x_k} (q^{y_k-x_k-N} z_k/b)_{x_k}} = \frac{q^{x_k-y_k}}{(q^{-N} z_k/b)_{y_k}}.$$

This gives

$$R = \frac{1}{\theta(q^{N+1})} \sum_{\substack{y_1 + \dots + y_n = N+1 \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j + y_k}} \right. \\ \times \prod_{k=1}^n \frac{\theta(q^{N+y_k} b z_k) q^{\binom{y_k}{2}} z_k^{y_k} \prod_{j=1}^{n-1} (z_k a_j, z_k / a_j)_{y_k}}{\theta(q^N b z_k) (b z_k, q^{-N} z_k / b)_{y_k} \prod_{j=1}^n (q z_k / z_j)_{y_k}} \\ \left. \times \sum_{\substack{x_1 + \dots + x_n = 1 \\ x_1, \dots, x_n \geq 0}} \prod_{k=1}^n \frac{(q^{y_k-1} b z_k)_{x_k} \prod_{j=1}^n (q^{y_k} z_k / z_j)_{x_k}}{(q^{N+y_k} b z_k)_{x_k} \prod_{j \neq k} (q^{y_k-y_j} z_k / z_j)_{x_k}} \right).$$

Here, the inner sum is identical to the one in (5.1) and thus equals the right-hand side of (5.2). It follows that

$$R = \sum_{\substack{y_1 + \dots + y_n = N+1 \\ y_1, \dots, y_n \geq 0}} \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j + y_k}} \\ \times \prod_{k=1}^n \frac{q^{\binom{y_k}{2}} z_k^{y_k} \prod_{j=1}^{n-1} (z_k a_j, z_k / a_j)_{y_k}}{(b z_k, q^{-N} z_k / b)_{y_k} \prod_{j=1}^n (q z_k / z_j)_{y_k}},$$

which completes the proof.  $\square$

Next we rewrite Theorem 6.1 in analogy with Corollary 5.2. This is, up to a change of variables, the form in which the case  $p = 0$  of Theorem 6.1 is given in [38].

**Corollary 6.2.** *The following identity holds:*

$$\sum_{\substack{y_1 + \dots + y_n \leq N \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k + |y|})}{\theta(az_k)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j + y_k}} \prod_{j,k=1}^n \frac{(z_k b_j, z_k / b_j)_{y_k}}{(q z_k / z_j)_{y_k}} \right. \\ \times \prod_{j=1}^n \frac{(az_j)_{|y|} (aq / z_j)_{|y| - y_j}}{(aq b_j, aq / b_j)_{|y|}} \frac{(q^{-N}, c, a^2 q^{N+1} / c)_{|y|}}{\prod_{k=1}^n (aq z_k / c, q^{-N} c z_k / a, q^{N+1} a z_k)_{y_k}} q^{|y|} \left. \right) \\ = \prod_{k=1}^n \frac{(aq z_k, aq / z_k, aq b_k / c, aq / b_k c)_N}{(aq z_k / c, aq / z_k c, aq b_k, aq / b_k)_N}.$$

With the same proof as for Corollary 5.3, one may obtain an accompanying identity for a sum supported on a hyper-rectangle. When  $p = 0$ , this identity was found independently by Schlosser [38, Theorem 5.14] and Bhatnagar [2, Theorem 7].



**Corollary 6.3.** *If  $a^2q = bcd$ , the following identity holds:*

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j+y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j, q^{m_j} z_j z_k)_{y_k}}{(q z_k / z_j)_{y_k}} \right. \\ & \times \prod_{j=1}^n \frac{(az_j)_{|y|} (aq/z_j)_{|y|-y_j}}{(aq^{1+m_j} z_j, aq^{1-m_j} / z_j)_{|y|}} \frac{(b, c, d)_{|y|}}{\prod_{k=1}^n (aq z_k / b, aq z_k / c, aq z_k / d)_{y_k}} q^{|y|} \Big) \\ & = \prod_{k=1}^n \frac{(aq z_k, bz_k / a, cz_k / a, dz_k / a)_{m_k}}{(z_k / a, aq z_k / b, aq z_k / c, aq z_k / d)_{m_k}}. \end{aligned}$$

Corollary 6.3 has been written in a form similar to how its special case  $p = 0$  is given in [38]. In [2], it is written down in a different way, which is obtained by reversing the order of summation. We give this version of Corollary 6.3 also in the elliptic case, since it is useful for applications, in particular for obtaining the Bailey-type transformations of Section 8.

**Corollary 6.4.** *If  $a^2q^{1+|m|} = bcde$ , the following identity holds:*

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \frac{\prod_{1 \leq j < k \leq n} (aq z_j z_k / e)_{y_j+y_k}}{\prod_{j,k=1}^n (aq z_j z_k / e)_{y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(q z_k / z_j)_{y_k}} \right. \\ & \times \prod_{j=1}^n \frac{(az_j, e/z_j)_{|y|}}{(aq^{1+m_j} z_j)_{|y|} (e/z_j)_{|y|-y_j}} \frac{\prod_{k=1}^n (bz_k, cz_k, dz_k)_{y_k}}{(aq/b, aq/c, aq/d)_{|y|}} q^{|y|} \Big) \\ & = \frac{\prod_{1 \leq j < k \leq n} (aq z_j z_k / e)_{m_j+m_k} \prod_{k=1}^n (aq z_k, aq z_k / be, aq z_k / ce, aq^{1+|m|-m_k} / bcz_k)_{m_k}}{\prod_{j,k=1}^n (aq z_j z_k / e)_{m_k} (aq/b, aq/c, aq/bce)_{|m|}}. \end{aligned}$$

Note that using (3.6) we may write

$$\frac{\prod_{k=1}^n (aq^{1+|m|-m_k} / bcz_k)_{m_k}}{(aq/bce)_{|m|}} = \frac{e^{|m|} q^{\sum_{j < k} m_j m_k}}{z_1^{m_1} \dots z_n^{m_n}} \frac{\prod_{k=1}^n (aq z_k / de)_{m_k}}{(aq/d)_{|m|}},$$

which makes the right-hand side appear more symmetric.

## 7. An elliptic $C_n$ Jackson summation

Next we give an elliptic Jackson summation on the root system  $C_n$ . In the case  $p = 0$  it was found independently by Denis and Gustafson [5, Theorem 4.1] and by Milne and Lilly [31, Theorem 6.13]. The general case was stated by van Diejen and Spiridonov [9], who showed that it follows from an elliptic Selberg integral conjectured in [8]. They also used modular forms to prove that the Taylor expansion in  $\log q$  of both sides agree up to order 10 (more precisely, this follows from the non-existence of cusp forms with weight less than 12).

**Theorem 7.1.** *If  $a^2 q^{1+|m|} = bcde$ , one has the identity*

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j \leq k \leq n} \frac{\theta(az_j z_k q^{y_j+y_k})}{\theta(az_j z_k)} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j, az_j z_k)_{y_k}}{(qz_k / z_j, aq^{1+m_j} z_j z_k)_{y_k}} \right. \\ & \times \left. \prod_{k=1}^n \frac{(bz_k, cz_k, dz_k, ez_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d, aqz_k/e)_{y_k}} q^{|y|} \right) \\ & = \frac{\prod_{j,k=1}^n (aqz_j z_k)_{m_k}}{\prod_{1 \leq j < k \leq n} (aqz_j z_k)_{m_j+m_k}} \frac{(aq/bc, aq/bd, aq/cd)_{|m|}}{\prod_{k=1}^n (aqz_k/b, aqz_k/c, aqz_k/d, q^{-m_k} e / az_k)_{m_k}}. \end{aligned}$$

The method that we have used above for the root systems  $A_n$  and  $D_n$  fails for this  $C_n$  identity. The reason is that the sum does not involve any factors of the form  $(b)_{|y|}$ , so we cannot convert it to a sum on a simplex  $|y| \leq N$ . However, the absence of such factors allows one to employ a different method, namely, determinant evaluation.

The idea to obtain multivariable hypergeometric summations from determinant evaluations is due to Gustafson and Krattenthaler [21] and was further developed by Schlosser [39]. Previously, no close relation has been known between sums coming from determinant evaluations and the Gustafson–Milne-type sums that are our main concern here. It may therefore seem surprising that we will obtain Theorem 7.1 as a special case of a multivariable Jackson sum proved by Warnaar [46] using determinant evaluation (his “first” summation mentioned in the introduction). In particular, it follows that the Denis–Gustafson–Milne–Lilly sum is a special case of the case  $p = 0$  of Warnaar’s identity, which is due to Schlosser [39, Theorem 4.2].

Warnaar’s first Jackson summation may be written as

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^N \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j \leq k \leq n} \frac{\theta(az_j z_k q^{y_j+y_k})}{\theta(az_j z_k)} \right. \\ & \times \left. \prod_{k=1}^n \frac{(az_k^2, bz_k, cz_k, dz_k, ez_k, q^{-N})_{y_k}}{(q, aqz_k/b, aqz_k/c, aqz_k/d, aqz_k/e, aq^{N+1} z_k^2)_{y_k}} q^{|y|} \right) \\ & = \prod_{1 \leq j < k \leq n} \frac{\theta(aq^N z_j z_k)}{\theta(az_j z_k)} \prod_{k=1}^n \frac{(aqz_k^2, aq^{2-k}/bc, aq^{2-k}/bd, aq^{2-k}/cd)_N}{(aqz_k/b, aqz_k/c, aqz_k/d, q^{-N} e / az_k)_N}, \quad (7.1) \end{aligned}$$

where  $a^2 q^{2+N-n} = bcde$ . In [35], we used the case  $N = 1$  of this identity to prove Warnaar’s conjectured “second” elliptic Jackson summation (different in nature both from (7.1) and from the results of the present paper). We also remarked that the case  $N = 1$  of (7.1) is equivalent to the case  $m_j \equiv 1$  of Theorem 7.1. We will prove Theorem 7.1 by combining this observation with the fact that the *general* case of Theorem 7.1 is equivalent to its special case when  $m_j \equiv 1$ . This may seem strange but is actually a common phenomenon: for instance, the summation formulas of Corollaries 5.3 and 6.3 are also equivalent to their special case  $m_j \equiv 1$ . In a somewhat different context, this phenomenon was exploited in the proof of Theorem 3.1 in [36].

**Proof of Theorem 7.1.** We first verify the claim made in [35] that the case  $N = 1$  of (7.1) is equivalent to the case  $m_j \equiv 1$  of Theorem 7.1.

If we let  $N = 1$  in (7.1) and use (3.6) we obtain

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^1 \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j < k \leq n} \frac{\theta(az_j z_k q^{y_j+y_k})}{\theta(az_j z_k)} \right. \\ & \times \prod_{k=1}^n \frac{(bz_k, cz_k, dz_k, ez_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d, aqz_k/e)_{y_k}} (-1)^{|y|} \Big) \\ & = \frac{\prod_{1 \leq j \leq k \leq n} \theta(aqz_j z_k)}{\prod_{1 \leq j < k \leq n} \theta(az_j z_k)} \prod_{k=1}^n \frac{\theta(aq^{2-k}/bc, aq^{2-k}/bd, aq^{2-k}/cd)}{\theta(aqz_k/b, aqz_k/c, aqz_k/d, q^{-1}e/az_k)}, \end{aligned} \quad (7.2)$$

where  $a^2 q^{3-n} = bcde$ . Next we put  $m_j \equiv 1$  in Theorem 7.1 and rewrite the left-hand side using the two identities

$$\begin{aligned} & \frac{\Delta(zq^y)}{\Delta(z)} \prod_{j,k=1}^n \frac{(q^{-1}z_k/z_j)_{y_k}}{(qz_k/z_j)_{y_k}} = (-1)^{|y|} q^{-|y|} \frac{\Delta(zq^{-y})}{\Delta(z)}, \\ & \prod_{1 \leq j \leq k \leq n} \frac{\theta(az_j z_k q^{y_j+y_k})}{\theta(az_j z_k)} \prod_{j,k=1}^n \frac{(az_j z_k)_{y_k}}{(aq^2 z_j z_k)_{y_k}} = \prod_{1 \leq j < k \leq n} \frac{\theta(az_j z_k q^{2-y_j-y_k})}{\theta(az_j z_k q^2)}. \end{aligned}$$

These are easily verified by writing

$$\prod_{j,k=1}^n a_{jk} = \prod_{k=1}^n a_{kk} \prod_{1 \leq j < k \leq n} a_{jk} a_{kj}$$

and then considering the four cases  $y_j, y_k = 0, 1$  separately. Similarly, on the right-hand side we write

$$\frac{\prod_{j,k=1}^n (aqz_j z_k)_1}{\prod_{1 \leq j < k \leq n} (aqz_j z_k)_2} = \frac{\prod_{1 \leq j \leq k \leq n} \theta(aqz_j z_k)}{\prod_{1 \leq j < k \leq n} \theta(aq^2 z_j z_k)}.$$

This gives

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^1 \left( \frac{\Delta(zq^{-y})}{\Delta(z)} \prod_{1 \leq j < k \leq n} \frac{\theta(az_j z_k q^{2-y_j-y_k})}{\theta(az_j z_k q^2)} \right. \\ & \times \prod_{k=1}^n \frac{(bz_k, cz_k, dz_k, ez_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d, aqz_k/e)_{y_k}} (-1)^{|y|} \Big) \\ & = \frac{\prod_{1 \leq j \leq k \leq n} \theta(aqz_j z_k)}{\prod_{1 \leq j < k \leq n} \theta(aq^2 z_j z_k)} \prod_{k=1}^n \frac{\theta(aq^k/bc, aq^k/bd, aq^k/cd)}{\theta(aqz_k/b, aqz_k/c, aqz_k/d, q^{-1}e/az_k)}, \end{aligned} \quad (7.3)$$

where  $a^2 q^{1+n} = bcde$ . After the change of variables  $a \mapsto aq^2$ ,  $q \mapsto q^{-1}$ , this is (7.2). (Here we use that, since  $y_k \in \{0, 1\}$ , the elliptic Pochhammer symbols occurring in (7.3) do not depend on  $q$  for their definition.)

Next we prove that (7.3) implies the general case of Theorem 7.1. For this we replace  $n$  in (7.3) by  $|m| = m_1 + \dots + m_n$  and denote the parameters  $z_j$  in (7.3) by  $w_j$ . We choose these parameters as (compare [36], where this type of choice turned up naturally)

$$(w_1, \dots, w_{|m|}) = (z_1, qz_1, \dots, q^{m_1-1}z_1, \dots, z_n, qz_n, \dots, q^{m_n-1}z_n). \quad (7.4)$$

Then the factor  $\Delta(wq^{-y})$  vanishes unless  $y$  is of the form

$$y = (\underbrace{1, \dots, 1, 0, \dots, 0}_{m_1}, \dots, \underbrace{1, \dots, 1, 0, \dots, 0}_{m_n}), \quad 0 \leq x_j \leq m_j. \quad (7.5)$$

We claim that if we rewrite (7.3) using the  $x_j$  as summation variables, we recover Theorem 7.1.

The single products in (7.3) are easily handled using the obvious identities

$$\prod_{k=1}^{|m|} (bw_k)_{y_k} = \prod_{k=1}^n (bz_k)_{x_k}, \quad \prod_{k=1}^{|m|} \theta(bw_k) = \prod_{k=1}^n (bz_k)_{m_k}.$$

As for the double products, we will prove that, for parameters related by (7.4) and (7.5),

$$\prod_{1 \leq j < k \leq |m|} \frac{\theta(aw_j w_k q^{2-y_j-y_k})}{\theta(aw_j w_k q^2)} = \prod_{1 \leq j \leq k \leq n} \frac{\theta(az_j z_k q^{x_j+x_k})}{\theta(az_j z_k)} \prod_{j,k=1}^n \frac{(az_j z_k)_{x_k}}{(aq^{1+m_j} z_j z_k)_{x_k}}, \quad (7.6)$$

$$\frac{\Delta(wq^{-y})}{\Delta(w)} = (-1)^{|x|} q^{|x|} \frac{\Delta(zq^x)}{\Delta(z)} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j)_{x_k}}{(qz_k / z_j)_{x_k}}, \quad (7.7)$$

$$\frac{\prod_{1 \leq j \leq k \leq |m|} \theta(aqw_j w_k)}{\prod_{1 \leq j < k \leq |m|} \theta(aq^2 w_j w_k)} = \frac{\prod_{j,k=1}^n (aqz_j z_k)_{m_k}}{\prod_{1 \leq j < k \leq n} (aqz_j z_k)_{m_j+m_k}}. \quad (7.8)$$

Assuming that these identities have been proved it is easily checked that (7.3) reduces to the identity in Theorem 7.1, thereby completing the proof.

The left-hand sides of (7.6) and (7.7) are both of the form

$$\prod_{1 \leq j < k \leq |m|} \frac{f(w_j q^{-y_j}, w_k q^{-y_k})}{f(w_j, w_k)}.$$

We decompose these products as  $ABC$ , where

$$A = \prod_{j,k=1}^n \prod_{u=1}^{x_k} \prod_{t=x_j+1}^{m_j} \frac{f(z_j q^{t-1}, z_k q^{u-2})}{f(z_j q^{t-1}, z_k q^{u-1})},$$

$$B = \prod_{k=1}^n \prod_{1 \leq t < u \leq x_k} \frac{f(z_k q^{t-2}, z_k q^{u-2})}{f(z_k q^{t-1}, z_k q^{u-1})},$$

$$C = \prod_{1 \leq j < k \leq n} \prod_{u=1}^{x_k} \prod_{t=1}^{x_j} \frac{f(z_j q^{t-2}, z_k q^{u-2})}{f(z_j q^{t-1}, z_k q^{u-1})}.$$

Here we use the symmetry and antisymmetry, respectively, of  $f$  to collect the factors with  $y_j \neq y_k$  as in  $A$ .

In the case of (7.6), we have

$$\begin{aligned} A &= \prod_{j,k=1}^n \prod_{u=1}^{x_k} \prod_{t=x_j+1}^{m_j} \frac{\theta(az_j z_k q^{t+u-1})}{\theta(az_j z_k q^{t+u})} = \prod_{j,k=1}^n \prod_{u=1}^{x_k} \frac{\theta(az_j z_k q^{x_j+u})}{\theta(az_j z_k q^{m_j+u})} \\ &= \prod_{j,k=1}^n \frac{(az_j z_k q^{1+x_j})_{x_k}}{(az_j z_k q^{1+m_j})_{x_k}}. \end{aligned} \quad (7.9)$$

The factor  $B$  may be computed using the elementary identity

$$\prod_{1 \leq j < k \leq n} \frac{f(j+k-2)}{f(j+k)} = \prod_{k=1}^n \frac{f(k)}{f(k+n-1)},$$

which gives

$$B = \prod_{k=1}^n \prod_{1 \leq t < u \leq x_k} \frac{\theta(az_k^2 q^{t+u-2})}{\theta(az_k^2 q^{t+u})} = \prod_{k=1}^n \frac{(az_k^2 q)_{x_k}}{(az_k^2 q^{x_k})_{x_k}}. \quad (7.10)$$

Finally,  $C$  may be computed similarly as  $A$ :

$$\begin{aligned} C &= \prod_{1 \leq j < k \leq n} \prod_{u=1}^{x_k} \prod_{t=1}^{x_j} \frac{\theta(az_j z_k q^{t+u-2})}{\theta(az_j z_k q^{t+u})} = \prod_{1 \leq j < k \leq n} \prod_{u=1}^{x_k} \frac{\theta(az_j z_k q^{u-1}, az_j z_k q^u)}{\theta(az_j z_k q^{x_j+u-1}, az_j z_k q^{x_j+u})} \\ &= \prod_{1 \leq j < k \leq n} \frac{(az_j z_k, az_j z_k q)_{x_k}}{(az_j z_k q^{x_j}, az_j z_k q^{1+x_j})_{x_k}}. \end{aligned} \quad (7.11)$$

Combining (7.9)–(7.11) gives

$$\prod_{1 \leq j < k \leq |m|} \frac{\theta(aw_j w_k q^{2-y_j-y_k})}{\theta(aw_j w_k q^2)} = \prod_{j,k=1}^n \frac{(az_j z_k q^{1+x_j})_{x_k}}{(az_j z_k q^{1+m_j})_{x_k}} \prod_{1 \leq j \leq k \leq n} \frac{(az_j z_k q)_{x_k}}{(az_j z_k q^{x_j})_{x_k}} \\ \times \prod_{1 \leq j < k \leq n} \frac{(az_j z_k)_{x_k}}{(az_j z_k q^{1+x_j})_{x_k}}.$$

To complete the proof of (7.6), we must show that

$$\prod_{j,k=1}^n \frac{(az_j z_k q^{1+x_j})_{x_k}}{(az_j z_k)_{x_k}} \prod_{1 \leq j \leq k \leq n} \frac{(az_j z_k q)_{x_k}}{(az_j z_k q^{x_j})_{x_k}} \prod_{1 \leq j < k \leq n} \frac{(az_j z_k)_{x_k}}{(az_j z_k q^{1+x_j})_{x_k}} \\ = \prod_{1 \leq j \leq k \leq n} \frac{\theta(az_j z_k q^{x_j+x_k})}{\theta(az_j z_k)},$$

which is easily verified after writing

$$\prod_{j,k=1}^n \frac{(az_j z_k q^{1+x_j})_{x_k}}{(az_j z_k)_{x_k}} = \prod_{1 \leq j < k \leq n} \frac{(az_j z_k q^{1+x_j})_{x_k}}{(az_j z_k)_{x_k}} \prod_{1 \leq j \leq k \leq n} \frac{(az_j z_k q^{1+x_k})_{x_j}}{(az_j z_k)_{x_j}}.$$

Eq. (7.7) may be proved in a similar way, and we do not give the details. The case  $p = 0$ , which is essentially the same, was done as part of the proof of Theorem 3.1 in [36].

To prove (7.8), we write the left-hand side as

$$\frac{\prod_{1 \leq j \leq k \leq |m|} \theta(aq w_j w_k)}{\prod_{1 \leq j < k \leq |m|} \theta(aq^2 w_j w_k)} = \prod_{k=1}^n \frac{\prod_{1 \leq t \leq u \leq m_k} \theta(aq^{t+u-1} z_k^2)}{\prod_{1 \leq t < u \leq m_k} \theta(aq^{t+u} z_k^2)} \\ \times \prod_{1 \leq j < k \leq n} \prod_{u=1}^{m_k} \prod_{t=1}^{m_j} \frac{\theta(aq^{t+u-1} z_j z_k)}{\theta(aq^{t+u} z_j z_k)}.$$

Here, the first part of the product may be computed using the identity

$$\frac{\prod_{1 \leq j \leq k \leq n} f(j+k-1)}{\prod_{1 \leq j < k \leq n} f(j+k)} = \prod_{k=1}^n f(k)$$

giving

$$\prod_{k=1}^n \frac{\prod_{1 \leq t \leq u \leq m_k} \theta(aq^{t+u-1} z_k^2)}{\prod_{1 \leq t < u \leq m_k} \theta(aq^{t+u} z_k^2)} = \prod_{k=1}^n (aq z_k^2)_{m_k}. \quad (7.12)$$

Finally, the second part is given by

$$\begin{aligned} \prod_{1 \leq j < k \leq n} \prod_{u=1}^{m_k} \prod_{t=1}^{m_j} \frac{\theta(aq^{t+u-1} z_j z_k)}{\theta(aq^{t+u} z_j z_k)} &= \prod_{1 \leq j < k \leq n} \prod_{u=1}^{m_k} \frac{\theta(aq^u z_j z_k)}{\theta(aq^{u+m_j} z_j z_k)} \\ &= \prod_{1 \leq j < k \leq n} \frac{(aq z_j z_k)_{m_k} (aq z_j z_k)_{m_j}}{(aq z_j z_k)_{m_j+m_k}} = \frac{1}{\prod_{k=1}^n (aq z_k^2)_{m_k}} \frac{\prod_{j,k=1}^n (aq z_j z_k)_{m_k}}{\prod_{1 \leq j < k \leq n} (aq z_j z_k)_{m_j+m_k}}. \end{aligned} \quad (7.13)$$

Combining (7.12) and (7.13) we obtain (7.8). This completes the proof of Theorem 7.1.  $\square$

## 8. Elliptic Bailey transformations

As applications of our multivariable elliptic Jackson summations, we may obtain a host of multivariable Bailey transformations for elliptic hypergeometric series on root systems. It should be observed that the proofs are almost identical to those for the case  $p=0$  given in [3,32], since they mainly involve manipulation of Pochhammer symbols using the elementary identities (3.4)–(3.6), which do not explicitly depend on  $p$ . The only difference is that one must replace polynomial continuation arguments by appealing to the quasi-periodicity, as in the proof of Corollary 5.3. There is therefore no need to give detailed proofs of these new elliptic Bailey transformations. Anyway, we sketch the proof of the first one to indicate to the reader what is involved; for the remaining ones we provide enough details so that the interested reader should be able to verify the calculations.

We begin with a Bailey transformation for elliptic hypergeometric series on  $A_n$ . When  $p=0$ , it is due independently to Denis and Gustafson [5, Theorem 3.1], who derived it by residue calculus from a multivariable integral transformation and to Milne and Newcomb [32, Theorem 3.1], who used the method sketched below. For the fact that the results of [5] and [32] are equivalent, as well as for a third proof, we refer to [36].

**Corollary 8.1.** *Assuming  $a^3 q^{2+|m|} = bcdefg$  and writing  $\lambda = a^2 q/bce$ , the following identity holds:*

$$\begin{aligned} &\sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \frac{(b, c, d)_{|y|}}{(aq/e, aq/f, aq/g)_{|y|}} \prod_{j=1}^n \frac{(az_j)_{|y|}}{(aq^{1+m_j} z_j)_{|y|}} q^{|y|} \right. \\ &\quad \times \prod_{k=1}^n \frac{(ez_k, fz_k, gz_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d)_{y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k/z_j)_{y_k}}{(qz_k/z_j)_{y_k}} \Big) \\ &= \left( \frac{a}{\lambda} \right)^{|m|} \frac{(\lambda q/f, \lambda q/g)_{|m|}}{(aq/f, aq/g)_{|m|}} \prod_{k=1}^n \frac{(aqz_k, \lambda qz_k/d)_{m_k}}{(\lambda qz_k, aqz_k/d)_{m_k}} \end{aligned}$$

$$\begin{aligned} & \times \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(\lambda z_k q^{y_k+|y|})}{\theta(\lambda z_k)} \frac{(\lambda b/a, \lambda c/a, d)_{|y|}}{(aq/e, \lambda q/f, \lambda q/g)_{|y|}} \prod_{j=1}^n \frac{(\lambda z_j)_{|y|}}{(\lambda q^{1+m_j} z_j)_{|y|}} q^{|y|} \right. \\ & \times \left. \prod_{k=1}^n \frac{(\lambda e z_k/a, f z_k, g z_k)_{y_k}}{(aq z_k/b, aq z_k/c, \lambda q z_k/d)_{y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k/z_j)_{y_k}}{(q z_k/z_j)_{y_k}} \right). \end{aligned}$$

**Proof.** Let  $W$  denote the sum on the left-hand side. We want to expand each term by identifying the factor

$$\frac{(b, c)_{|y|}}{(aq/e)_{|y|}} \prod_{k=1}^n \frac{(e z_k)_{y_k}}{(aq z_k/b, aq z_k/c)_{y_k}}$$

with a part of the right-hand side of (5.3). We must then replace the parameters  $(a, b, c, d, e, m_k)$  in Corollary 5.3 with  $(\lambda, \lambda b/a, \lambda c/a, \lambda e/a, aq^{|y|}, y_k)$ . This gives

$$\begin{aligned} W = & \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \frac{(d, a/\lambda)_{|y|}}{(aq/f, aq/g)_{|y|}} \prod_{j=1}^n \frac{(az_j)_{|y|}}{(aq^{1+m_j} z_j)_{|y|}} q^{|y|} \right. \\ & \times \prod_{k=1}^n \frac{(f z_k, g z_k)_{y_k}}{(aq z_k/d, \lambda q z_k)_{y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k/z_j)_{y_k}}{(q z_k/z_j)_{y_k}} \\ & \times \sum_{x_1, \dots, x_n=0}^{y_1, \dots, y_n} \left\{ \frac{\Delta(zq^x)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(\lambda z_k q^{x_k+|x|})}{\theta(\lambda z_k)} \frac{(\lambda b/a, \lambda c/a)_{|x|}}{(aq/e, q^{1-|y|} \lambda/a)_{|x|}} \prod_{j=1}^n \frac{(\lambda z_j)_{|x|}}{(\lambda q^{1+y_j} z_j)_{|x|}} q^{|x|} \right. \\ & \times \left. \prod_{k=1}^n \frac{(\lambda e z_k/a, aq^{|y|} z_k)_{x_k}}{(aq z_k/b, aq z_k/c)_{x_k}} \prod_{j,k=1}^n \frac{(q^{-y_j} z_k/z_j)_{x_k}}{(q z_k/z_j)_{x_k}} \right\} \Bigg). \end{aligned}$$

Replacing  $y$  by  $y + x$  and changing the order of summation gives, after some elementary manipulations and an application of (3.8),

$$\begin{aligned} W = & \sum_{x_1, \dots, x_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^x)}{\Delta(z)} \frac{(\lambda b/a, \lambda c/a, d)_{|x|}}{(aq/e, aq/f, aq/g)_{|x|}} \prod_{j=1}^n \frac{(aq z_j)_{|x|+x_j}}{(aq^{1+m_j} z_j)_{|x|} (\lambda q^{|x|} z_j)_{x_j}} q^{|x|} \right. \\ & \times \left( \frac{a}{\lambda} \right)^{|x|} \prod_{k=1}^n \frac{(\lambda e z_k/a, f z_k, g z_k)_{x_k}}{(aq z_k/b, aq z_k/c, aq z_k/d)_{x_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k/z_j)_{x_k}}{(q z_k/z_j)_{x_k}} \\ & \times \sum_{y_1, \dots, y_n=0}^{m_1-x_1, \dots, m_n-x_n} \left\{ \frac{\Delta(zq^{x+y})}{\Delta(zq^x)} \prod_{k=1}^n \frac{\theta(az_k q^{x_k+|x|+y_k+|y|})}{\theta(az_k q^{x_k+|x|})} \frac{(dq^{|x|}, a/\lambda)_{|y|}}{(aq^{1+|x|}/f, aq^{1+|x|}/g)_{|y|}} q^{|y|} \right. \\ & \times \left. \prod_{j=1}^n \frac{(aq^{|x|+x_j} z_j)_{|y|}}{(aq^{1+|x|+m_j} z_j)_{|y|}} \prod_{k=1}^n \frac{(f q^{x_k} z_k, g q^{x_k} z_k)_{y_k}}{(aq^{1+x_k} z_k/d, \lambda q^{1+|x|+x_k} z_k)_{y_k}} \prod_{j,k=1}^n \frac{(q^{x_k-m_j} z_k/z_j)_{y_k}}{(q^{1+x_k-x_j} z_k/z_j)_{y_k}} \right\} \Bigg). \end{aligned}$$



The condition  $bcdefg = a^3 q^{2+|m|}$  means that the inner sum is as in Corollary 5.3, with  $(a, b, c, d, e, m_k, z_k)$  replaced by  $(aq^{|x|}, dq^{|x|}, a/\lambda, f, g, m_k - x_x, q^{x_k} z_k)$ . Using (3.5) and the fact that  $dfg = a\lambda q^{1+|m|}$ , the corresponding right-hand side of (5.3) may be written as

$$\left(\frac{a}{\lambda}\right)^{|m|-|x|} \frac{(\lambda q^{1+|x|}/f, \lambda q^{1+|x|}/g)_{|m|-|x|}}{(aq^{1+|x|}/f, aq^{1+|x|}/g)_{|m|-|x|}} \prod_{k=1}^n \frac{(aq^{1+|x|+x_k} z_k, \lambda q^{1+x_k} z_k/d)_{m_k-x_k}}{(\lambda q^{1+|x|+x_k} z_k, aq^{1+x_k} z_k/d)_{m_k-x_k}}.$$

Some further elementary manipulations completes the proof.  $\square$

There is also a version of Corollary 8.1 for series supported on a simplex. It can be obtained either by a slight modification of the proof of Corollary 8.1, using first Corollary 5.3 and then Corollary 5.2, instead of using Corollary 5.3 twice, or as a consequence of Corollary 8.1, using a continuation argument similar to the proof of Corollary 5.3. When  $p = 0$ , the resulting identity is [32, Theorem 3.3].

**Corollary 8.2.** Assuming  $a^3 q^{2+N} = b_1 \cdots b_{n+2} c d e z_1 \cdots z_n$  and writing  $\lambda = a^2 q / c d e$ , the following identity holds:

$$\begin{aligned} & \sum_{\substack{y_1 + \cdots + y_n \leq N \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \frac{(q^{-N}, c, d)_{|y|} \prod_{j=1}^n (az_j)_{|y|}}{(aq/e)_{|y|} \prod_{j=1}^{n+2} (aq/b_j)_{|y|}} q^{|y|} \right. \\ & \quad \times \prod_{k=1}^n \frac{(ez_k)_{y_k} \prod_{j=1}^{n+2} (b_j z_k)_{y_k}}{(aq^{1+N} z_k, aq z_k / c, aq z_k / d)_{y_k} \prod_{j=1}^n (q z_k / z_j)_{y_k}} \Big) \\ &= \left(\frac{a}{\lambda}\right)^N \prod_{k=1}^n \frac{(aq z_k)_N \prod_{j=1}^{n+2} (\lambda q / b_j)_N}{(\lambda q z_k)_N \prod_{j=1}^{n+2} (aq / b_j)_N} \\ & \quad \times \sum_{\substack{y_1 + \cdots + y_n \leq N \\ y_1, \dots, y_n \geq 0}} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(\lambda z_k q^{y_k+|y|})}{\theta(\lambda z_k)} \frac{(q^{-N}, \lambda c / a, \lambda d / a)_{|y|} \prod_{j=1}^n (\lambda z_j)_{|y|}}{(aq/e)_{|y|} \prod_{j=1}^{n+2} (\lambda q / b_j)_{|y|}} q^{|y|} \right. \\ & \quad \times \prod_{k=1}^n \frac{(\lambda e z_k / a)_{y_k} \prod_{j=1}^{n+2} (b_j z_k)_{y_k}}{(\lambda q^{1+N} z_k, aq z_k / c, aq z_k / d)_{y_k} \prod_{j=1}^n (q z_k / z_j)_{y_k}} \Big). \end{aligned}$$

In [3] a number of  $C_n$  and  $D_n$  Bailey transformations were obtained by judiciously combining  $A_n$ ,  $C_n$  and  $D_n$  Jackson summations, similarly as in the proof of Corollary 8.1. Starting from the elliptic Jackson summations obtained in the present paper, the same method yields elliptic  $C_n$  and  $D_n$  Bailey transformations. The seven transformations given in [3] fall into three groups. We are content with writing down one representative from each group explicitly.

Combining a  $C_n$  and a  $D_n$  Jackson summation, one may prove the following identity relating a  $C_n$  sum and an  $A_n$  sum. For  $p = 0$  it is Theorem 2.1 of [3].

**Corollary 8.3.** *Assuming that  $a^3 q^{2+|m|} = bcdefg$  and writing  $\lambda = a^2 q/bcd$ , the following identity holds:*

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{1 \leq j \leq k \leq n} \frac{\theta(az_j z_k q^{y_j+y_k})}{\theta(az_j z_k)} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j, az_j z_k)_{y_k}}{(qz_k / z_j, aq^{1+m_j} z_j z_k)_{y_k}} \right. \\ & \quad \times \left. \prod_{k=1}^n \frac{(bz_k, cz_k, dz_k, ez_k, fz_k, gz_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d, aqz_k/e, aqz_k/f, aqz_k/g)_{y_k}} q^{|y|} \right) \\ &= \frac{\prod_{j,k=1}^n (aqz_j z_k)_{m_k}}{\prod_{1 \leq j < k \leq n} (aqz_j z_k)_{m_j+m_k}} \frac{(\lambda q/e, \lambda q/f, aq/ef)_{|m|}}{\prod_{k=1}^n (\lambda qz_k, aqz_k/e, aqz_k/f, q^{-m_k} g/az_k)_{m_k}} \\ & \quad \times \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(\lambda z_k q^{y_k+|y|})}{\theta(\lambda z_k)} \frac{(\lambda b/a, \lambda c/a, \lambda d/a)_{|y|}}{(\lambda q/e, \lambda q/f, \lambda q/g)_{|y|}} \prod_{j=1}^n \frac{(\lambda z_j)_{|y|}}{(\lambda q^{1+m_j} z_j)_{|y|}} q^{|y|} \right. \\ & \quad \times \left. \prod_{k=1}^n \frac{(ez_k, fz_k, gz_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d)_{y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j)_{y_k}}{(qz_k / z_j)_{y_k}} \right). \end{aligned}$$

Using (3.6) one may write

$$\frac{(aq/ef)_{|m|}}{\prod_{k=1}^n (q^{-m_k} g/az_k)_{m_k}} = \left( \frac{a}{\lambda} \right)^{|m|} q^{-\sum_{j < k} m_j m_k} z_1^{m_1} \dots z_n^{m_n} \frac{(\lambda q/g)_{|m|}}{\prod_{k=1}^n (aqz_k/g)_{m_k}},$$

which makes the right-hand side appear more symmetric. Note also that, since the left-hand side of Corollary 8.3 is invariant under interchanging  $d$  and  $g$ , this must be true also for the right-hand side. This fact is equivalent to Corollary 8.1, which may thus alternatively be derived as a consequence of Corollary 8.3.

To prove Corollary 8.3, one may use the case

$$(a, b, c, d, z_k, m_k) \mapsto (\lambda/\sqrt{a}, \lambda b/a, \lambda c/a, \lambda d/a, \sqrt{a} z_k, y_k)$$

of Corollary 6.3 to expand the factor

$$\prod_{k=1}^n \frac{(bz_k, cz_k, dz_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d)_{y_k}}$$

on the left-hand side. After changing the order of summation as in the proof of Corollary 8.1, the inner sum may be computed using Theorem 7.1.

Next we give a  $D_n$  Bailey transformation that may be obtained by using a  $D_n$  Jackson summation twice. For  $p = 0$  it is equivalent to [3, Theorem 3.1].

**Corollary 8.4.** *Assuming  $a^3q^2 = bcdef$  and writing  $\lambda = a^2q/bcd$ , the following identity holds:*

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \prod_{j=1}^n \frac{(az_j, d/z_j)_{|y|} (aq/z_j)_{|y|-y_j}}{(aq^{1+m_j} z_j, aq^{1-m_j}/z_j)_{|y|} (d/z_j)_{|y|-y_j}} \right. \\ & \quad \times \prod_{j,k=1}^n \frac{(q^{-m_j} z_k/z_j, q^{m_j} z_j z_k)_{y_k}}{(qz_k/z_j, aqz_j z_k/d)_{y_k}} \prod_{1 \leq j < k \leq n} \frac{(aqz_j z_k/d)_{y_j+y_k}}{(z_j z_k)_{y_j+y_k}} \\ & \quad \times \prod_{k=1}^n \frac{(bz_k, cz_k)_{y_k}}{(aqz_k/e, aqz_k/f)_{y_k}} \frac{(e, f)_{|y|}}{(aq/b, aq/c)_{|y|}} q^{|y|} \Big) \\ &= \prod_{k=1}^n \frac{(aqz_k, z_k/\lambda, \lambda qz_k/e, \lambda qz_k/f)_{m_k}}{(\lambda qz_k, z_k/a, aqz_k/e, aqz_k/f)_{m_k}} \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(\lambda z_k q^{y_k+|y|})}{\theta(\lambda z_k)} \right. \\ & \quad \times \prod_{j=1}^n \frac{(\lambda z_j, \lambda d/az_j)_{|y|} (\lambda q/z_j)_{|y|-y_j}}{(\lambda q^{1+m_j} z_j, \lambda q^{1-m_j}/z_j)_{|y|} (\lambda d/az_j)_{|y|-y_j}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k/z_j, q^{m_j} z_j z_k)_{y_k}}{(qz_k/z_j, aqz_j z_k/d)_{y_k}} \\ & \quad \times \prod_{1 \leq j < k \leq n} \frac{(aqz_j z_k/d)_{y_j+y_k}}{(z_j z_k)_{y_j+y_k}} \prod_{k=1}^n \frac{(\lambda bz_k/a, \lambda cz_k/a)_{y_k}}{(\lambda qz_k/e, \lambda qz_k/f)_{y_k}} \frac{(e, f)_{|y|}}{(aq/b, aq/c)_{|y|}} q^{|y|} \Big). \end{aligned}$$

At a first glance, [3, Theorem 3.1] appears to have one more free parameter than Corollary 8.4, but, as noted in [3, Remark 3.3], one may specialize one parameter in that result without loss of generality.

To prove Corollary 8.4 one may use the case

$$(a, b, c, d, e, m_k) \mapsto (\lambda, \lambda b/a, \lambda c/a, aq^{|y|}, \lambda d/a, y_k)$$

of Corollary 6.4 to expand the factor

$$\frac{\prod_{1 \leq j < k \leq n} (aqz_j z_k/d)_{y_j+y_k}}{\prod_{j,k=1}^n (aqz_j z_k/d)_{y_k}} \frac{\prod_{k=1}^n (bz_k, cz_k, dq^{|y|-y_k}/z_k)_{y_k}}{(aq/b, aq/c)_{|y|}}.$$

Proceeding as in the proof of Corollary 8.1 calls for an application of Corollary 6.3 in the last step.

By a continuation argument, one may obtain a companion identity to Corollary 8.4 where the sum is over a simplex  $|y| \leq N$ . We do not write it out explicitly; the case  $p = 0$  is Theorem 3.7 of [3].

Another class of  $D_n$  Bailey transformations may be obtained by combining an  $A_n$  and a  $D_n$  Jackson summation. An example is the following identity, which is equivalent to Theorem 3.13 of [3] when  $p = 0$ .

**Corollary 8.5.** *Assuming that  $a^3 q^2 = bcdef$  and writing  $\lambda = a^2 q / bcf$ , the following identity holds:*

$$\begin{aligned} & \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(az_k q^{y_k+|y|})}{\theta(az_k)} \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j+y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j, q^{m_j} z_j z_k)_{y_k}}{(qz_k / z_j)_{y_k}} \right. \\ & \quad \times \prod_{j=1}^n \frac{(az_j)_{|y|} (aq/z_j)_{|y|-y_j}}{(aq^{1+m_j} z_j, aq^{1-m_j} / z_j)_{|y|}} \frac{(b, c, d, e)_{|y|}}{(aq/f)_{|y|}} \prod_{k=1}^n \frac{(fz_k)_{y_k}}{(aqz_k/b, aqz_k/c, aqz_k/d, aqz_k/e)_{y_k}} q^{|y|} \Big) \\ &= \prod_{k=1}^n \frac{(aqz_k, z_k/\lambda, \lambda qz_k/d, \lambda qz_k/e)_{m_k}}{(\lambda qz_k, z_k/a, aqz_k/d, aqz_k/e)_{m_k}} \sum_{y_1, \dots, y_n=0}^{m_1, \dots, m_n} \left( \frac{\Delta(zq^y)}{\Delta(z)} \prod_{k=1}^n \frac{\theta(\lambda z_k q^{y_k+|y|})}{\theta(\lambda z_k)} \right. \\ & \quad \times \prod_{1 \leq j < k \leq n} \frac{1}{(z_j z_k)_{y_j+y_k}} \prod_{j,k=1}^n \frac{(q^{-m_j} z_k / z_j, q^{m_j} z_j z_k)_{y_k}}{(qz_k / z_j)_{y_k}} \prod_{j=1}^n \frac{(\lambda z_j)_{|y|} (\lambda q/z_j)_{|y|-y_j}}{(\lambda q^{1+m_j} z_j, \lambda q^{1-m_j} / z_j)_{|y|}} \\ & \quad \times \frac{(\lambda b/a, \lambda c/a, d, e)_{|y|}}{(aq/f)_{|y|}} \prod_{k=1}^n \frac{(\lambda fz_k/a)_{y_k}}{(aqz_k/b, aqz_k/c, \lambda qz_k/d, \lambda qz_k/e)_{y_k}} q^{|y|} \Big). \end{aligned}$$

To prove this one may use the case  $(a, b, c, d, e, m_k) \mapsto (\lambda, \lambda b/a, \lambda c/a, \lambda f/a, aq^{|y|}, y_k)$  of Corollary 5.3 to expand the factor

$$\frac{(b, c)_{|y|}}{(aq/f)_{|y|}} \prod_{k=1}^n \frac{(fz_k)_{y_k}}{(aqz_k/b, aqz_k/c)_{y_k}}$$

on the left-hand side. The same method as before leads to a sum that is computed by Corollary 6.3.

In [3], three companion identities to the case  $p = 0$  of Corollary 8.5 are given. One of these [3, Theorem 3.9] is the equivalent identity obtained by reversing the order of summation, the other two [3, Theorems 3.11 and 3.16] (which are not equivalent) are continuations of the first two to the simplex. Again, these three identities are easily extended to the elliptic case, but we do not write them out explicitly.

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and 6.3 and, more importantly, verified that both sides are Jacobi modular forms in the sense of [10]. This is another strong indication that these identities are “natural”. I thank Prof. Spiridonov for this information, as well as for providing me with copies of the papers [41,42]. I also thank George Gasper for pointing out Ref. [45], as well as Michael Schlosser and the referee for some useful comments.

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